ME 321: Fluid Mechanics-I

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Lecture - 07 (14/06/2025)
Fluid Dynamics: Linear Momentum Equation

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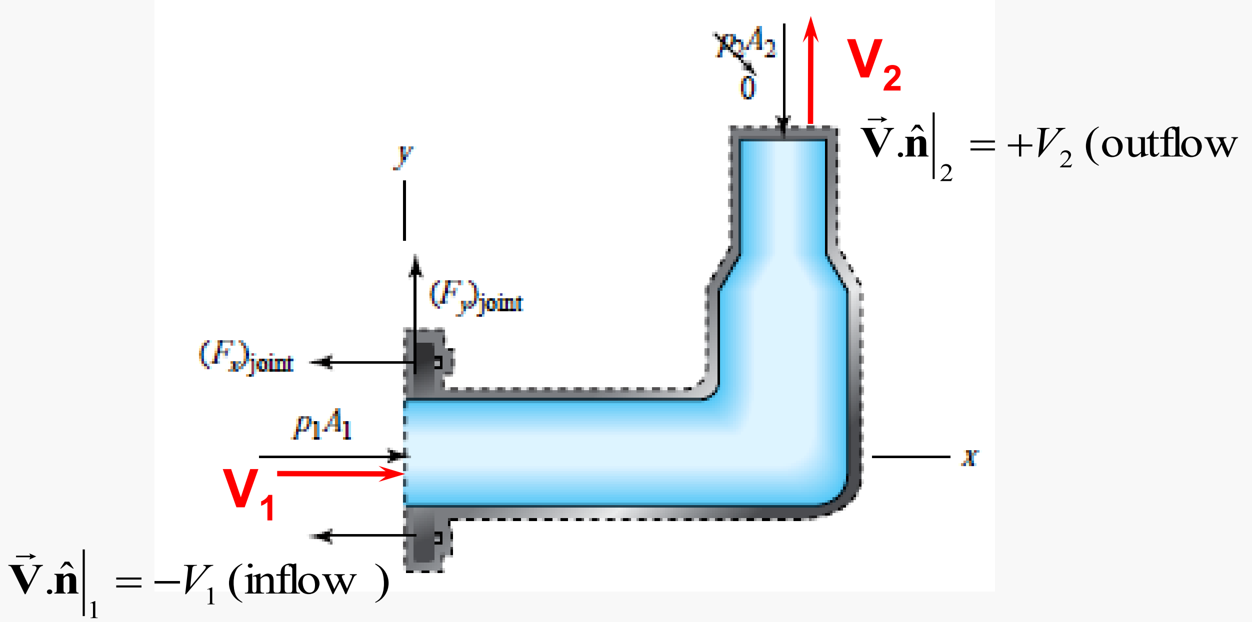
Recap: Momentum Principle

The above expression could be simplified considerably if a flow system has **only one entrance and one exit and if the flow is steady**:

$$\sum \vec{F}_{\mathrm{CV}} = \cancel{\frac{d}{dt}\int_{\mathrm{CV}} \vec{\mathbf{V}}\rho\, d\forall} + \int_{\mathrm{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \sum \vec{F}_{\mathrm{CV}} = \int_{\mathrm{CS}} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA$$

$$\Rightarrow \sum \vec{F}_{\mathrm{CV}} = \rho_2 A_2 V_2 \vec{\mathbf{V}}_2 - \rho_1 A_1 V_1 \vec{\mathbf{V}}_1$$



Using continuity:

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad \text{(mass flow rate)}$$

Then:

$$\boxed{\sum \vec{F}_{\mathrm{CV}} = \dot{m}\left(\vec{\mathbf{V}}_2 - \vec{\mathbf{V}}_1\right)}$$

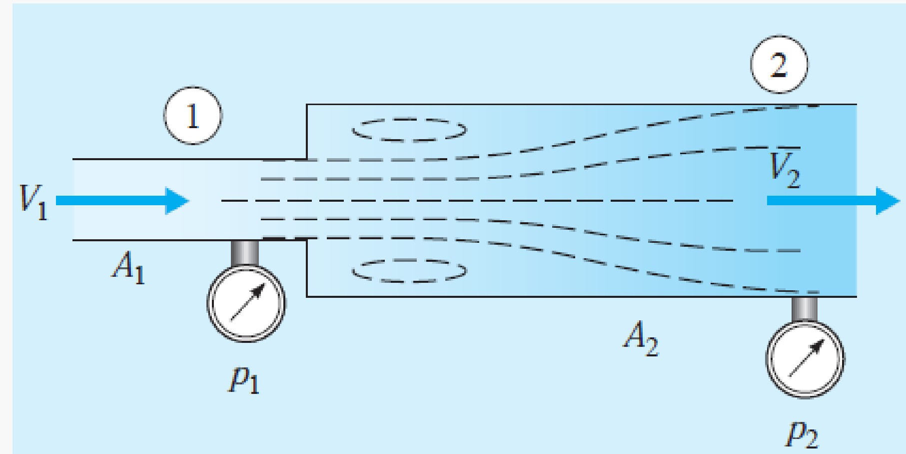
Note that the momentum equation is a vector equation which represents three scalar equations:

$$\boxed{\begin{aligned} x:&\ \sum F_x = \dot{m}\left(V_{2x} - V_{1x}\right)\\ y:&\ \sum F_y = \dot{m}\left(V_{2y} - V_{1y}\right)\\ z:&\ \sum F_z = \dot{m}\left(V_{2z} - V_{1z}\right)\end{aligned}}$$

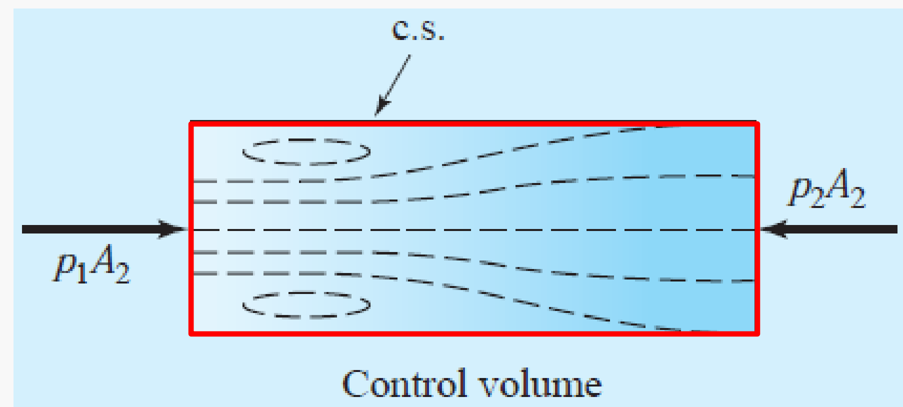
This is the fundamental principle which drives the turbomachinery (propulsion nozzle in jet engine, turbine, compressor cascade etc.)

Problem # 8 (Use of momentum equation to calculate head loss in sudden expansion)

Find an expression for the head loss in a sudden expansion in a pipe in terms of V_1 and the area ratio. Assume uniform velocity profiles and assume that the pressure at the sudden enlargement is p_1.



CV



Control volume

Solution:

In sudden expansion, the diameter changes from d_1 to d_2 ($d_2 > d_1$). The pressure at the sudden enlargement is closest to p_1 since the streamlines are approximately parallel as shown (there is no pressure variation normal to parallel streamlines); they take some distance to again fill the pipe. Hence the force acting on the left end of the control volume shown is approximately p_1A_2.

from steady flow momentum equation:

$$\boxed{\sum \vec{F}_{CV} = \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\rightarrow x: \sum F_x = \dot{m}\left(V_{2x} - V_{1x}\right)$$

$$\Rightarrow p_1A_2 - p_2A_2 = \rho A_2 V_2\left(V_2 - V_1\right)$$

$$\Rightarrow \frac{p_1 - p_2}{\rho} = V_2\left(V_2 - V_1\right)$$

Problem # 8 (Use of momentum equation to calculate head loss in sudden expansion)

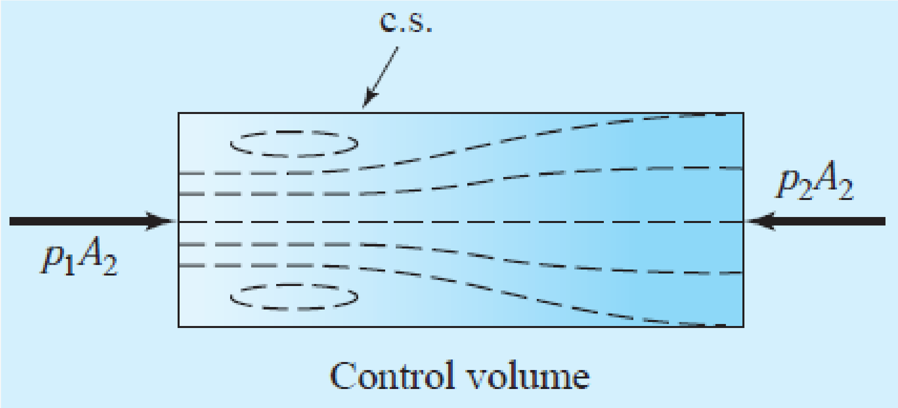
Using Bernoulli equation (energy equation)* considering head loss due to sudden expansion:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_L$$

$$\rightarrow h_L = \frac{p_1 - p_2}{\gamma} + \frac{V_1^2 - V_2^2}{2g} \qquad ; Z_1 = Z_2$$

$$= \frac{V_2(V_2 - V_1)}{g} - \frac{(V_2 + V_1)(V_2 - V_1)}{2g}$$

$$= \frac{(V_1 - V_2)^2}{2g}$$



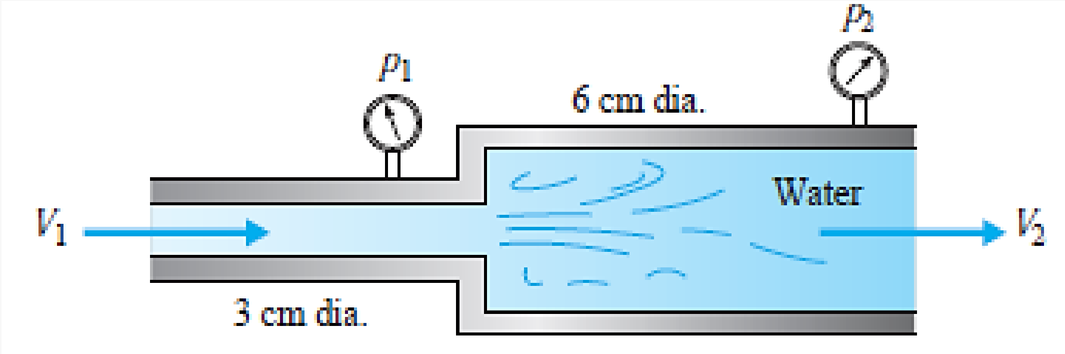
From continuity: $Q_1 = Q_2 \qquad \rightarrow V_1 A_1 = V_2 A_2 \qquad \rightarrow V_2 = \frac{A_1}{A_2} V_1$

$$\boxed{h_L = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{V_1^2}{2g}}$$

derivation to be covered in next classes.

Problem # 9

For the system shown in Figure, estimate the downstream pressure p_2 if p_1 60 kPa and $V_1 =$ 20 m/s. Also calculate the associated head loss. (*Note:* The pressure immediately after the pipe expansion is p_1.)



Solution:

$$A_1V_1 = A_2V_2$$

$$3^2(20) = 6^2V_2 \quad \rightarrow V_2 = 5\ m/s$$

from steady flow momentum equation for sudden expansion:

$$\rightarrow x : \sum F_x = \dot{m}(V_{2x} - V_{1x})$$

$$\Rightarrow p_1A_2 - p_2A_2 = \rho A_2V_2(V_2 - V_1)$$

$$\Rightarrow \frac{p_1 - p_2}{\rho} = V_2(V_2 - V_1)$$

$$p_2 = 135\ kPa$$

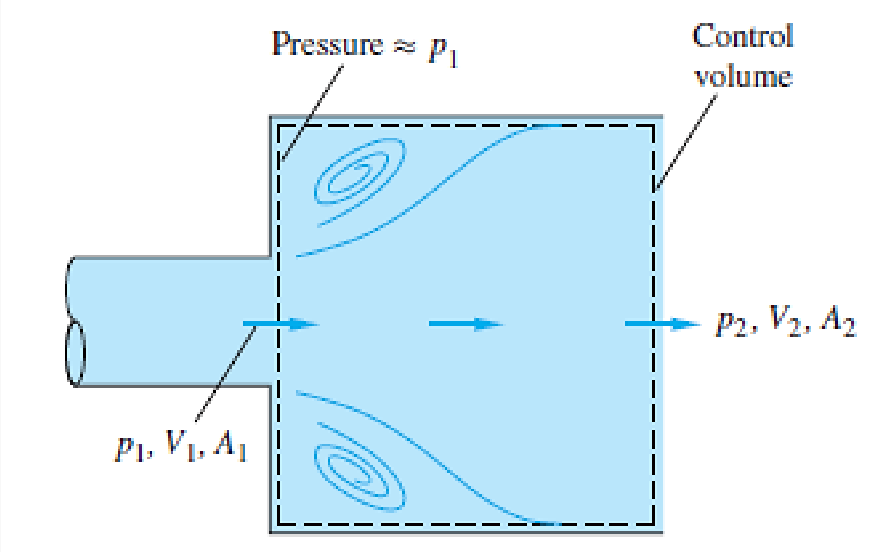
Associated head loss in sudden expansion flow:

$$\rightarrow h_L = \frac{(V_1 - V_2)^2}{2g}$$ = 11.47 m of Water (112.5 kPa)

Problem # 10

When a pipe flow suddenly expands from A_1 to A_2 , as in Fig, low-speed, low-friction eddies appear in the corners and the flow gradually expands to A_2 downstream. Using the suggested control volume for incompressible steady flow and assuming that $p \approx p_1$ on the corner annular ring as shown, show that the downstream pressure is given by (neglect wall friction)

$$p_2 = p_1 + \rho V_1^2 \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right)$$



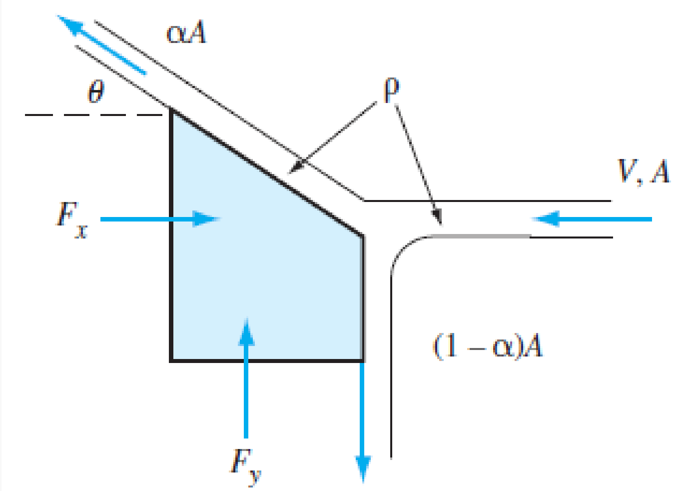
Problem # 11

A liquid jet of density ρ and area A strikes a block and splits into two jets, as shown in the figure. All three jets have the same velocity V. The upper jet exits at angle θ and area αA, the lower jet turns down at 90° and area $(1 - \alpha)$A.

(a) Derive a formula for the forces (F_x, F_y) required to support the block against momentum changes.

(b) Show that $F_y = 0$ only if $\alpha = 0.5$.

(c) Find the values of α and θ for which both F_x and F_y are zero.



Solution:

$$\sum F_x = F_x = \alpha\dot{m}(-V\cos\theta) - \dot{m}(-V) \quad \textit{where } \dot{m} = \rho AV \textit{ of the inlet jet}$$

$$\sum F_y = F_y = \alpha\dot{m}V\sin\theta + (1-\alpha)\dot{m}(-V)$$

Clean this up for the final result:

$$F_x = \dot{\mathbf{m}}\mathbf{V}(\mathbf{1} - \alpha\,\mathbf{cos}\,\theta)$$

$$F_y = \dot{\mathbf{m}}\mathbf{V}(\alpha\,\mathbf{sin}\,\theta + \alpha - \mathbf{1}) \quad \textit{Ans.}\ (a)$$

$$\mathrm{F_x} = \mathrm{F_y} = 0 \quad \text{only if: } \alpha = \mathbf{1},\ \theta = \mathbf{0°} \quad \textit{Ans.}\ (c)$$

Momentum equation applied to stationary/moving Deflectors/ Vanes

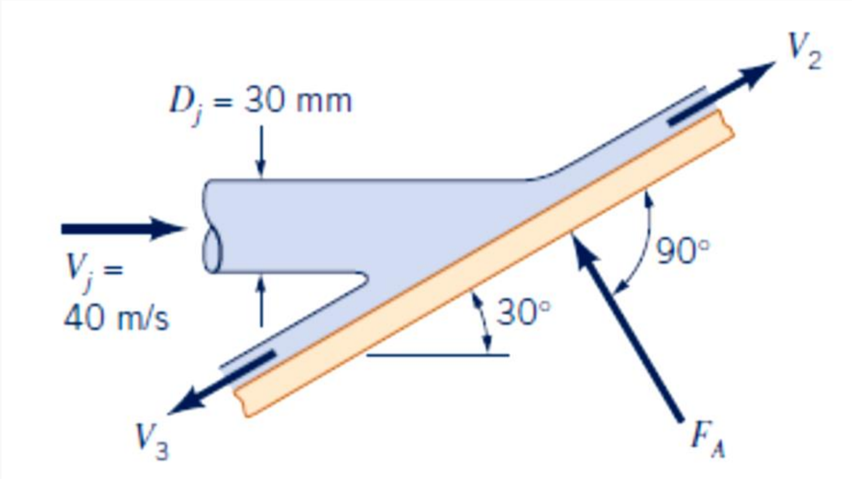
Problem # 12

A horizontal circular jet of air strikes a stationary flat plate as indicated in figure. The jet velocity is 40 m/s and the jet diameter is 30 mm. If the air velocity magnitude remains constant as the air flows over the plate in the directions shown, determine:

(a) the magnitude of F_A, the anchoring force required to hold the plate stationary,

(b) The fraction of mass flow along the plate surface in each of the two directions shown,

(c) the magnitude of F_A, the anchoring force required to allow the plate to move to right at a constant speed of 10 m/s.



Solution:

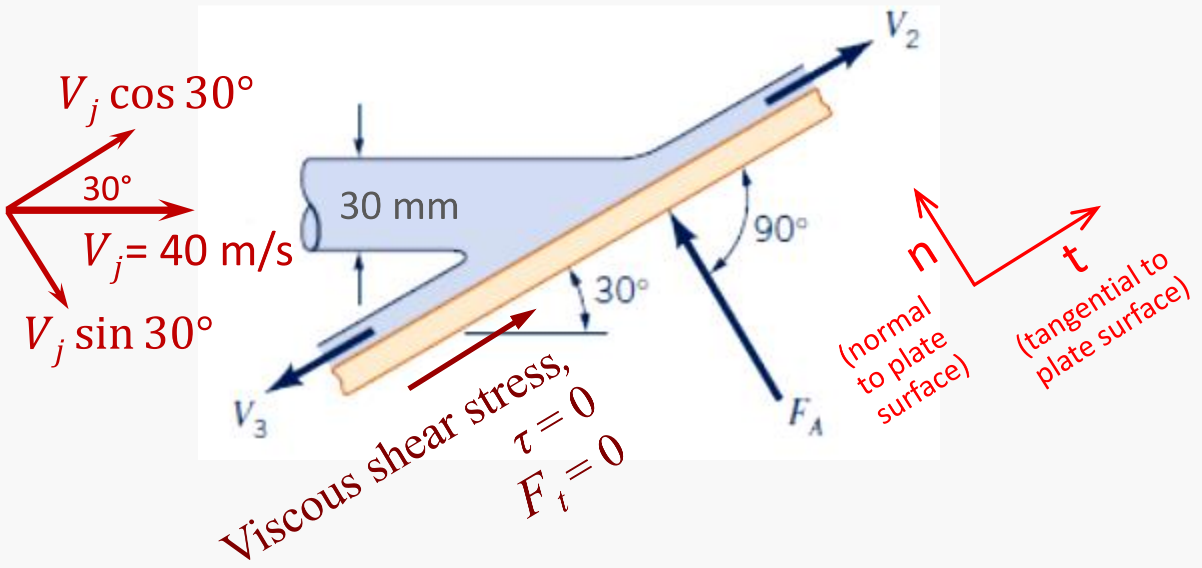
Steady flow momentum equation normal to the plate (n):

$$\sum F_n = (\dot{m}v)_{out} - (\dot{m}v)_{in}$$

$$\rightarrow F_A = 0 - \dot{m}_j(-v_j \sin 30°)$$; outflow momentum along tangential direction

$$\rightarrow F_A = \left(\rho \frac{\pi}{4} D_j^{\,2} v_j\right)(v_j \sin 30°)$$

F_A = 0.696 N <u>Ans. (a)</u>

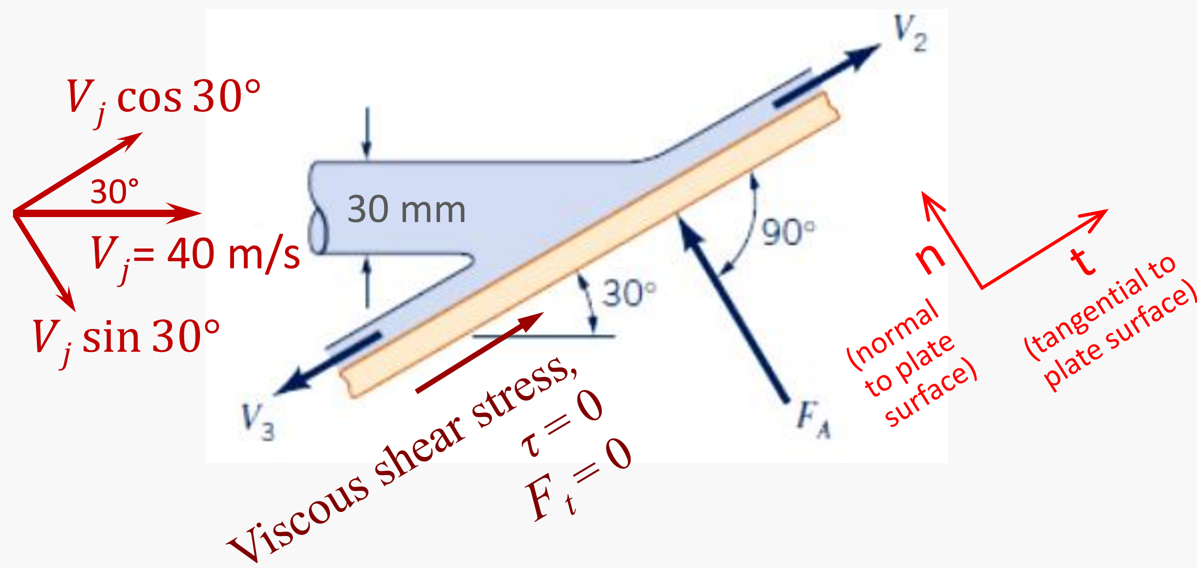


Problem # 12

(b) Steady flow momentum equation tangential to the plate (t):

$$\sum F_t = (\dot{m}v)_{out} - (\dot{m}v)_{in}$$

$$\rightarrow 0 = [\dot{m}_2 v_2 + \dot{m}_3(-v_3)] - \dot{m}_j(v_j \cos 30°)$$



Since the air velocity magnitude remains constant, i.e. $v_2 = v_3 = v_j$ ($p_2 = p_3 = p_j$ (free surface) and neglecting the elevation difference); there are no viscous effect along the plate surface which gives $F_t = 0$.

$$\dot{m}_3 = \dot{m}_2 - \dot{m}_j \sin 30°$$

Again, from conservation of mass:

$$\dot{m}_j = \dot{m}_2 + \dot{m}_3$$

On solving the above two equations

$$\dot{m}_2 = 0.933\,\dot{m}_j$$
$$\dot{m}_3 = 0.067\,\dot{m}_j$$

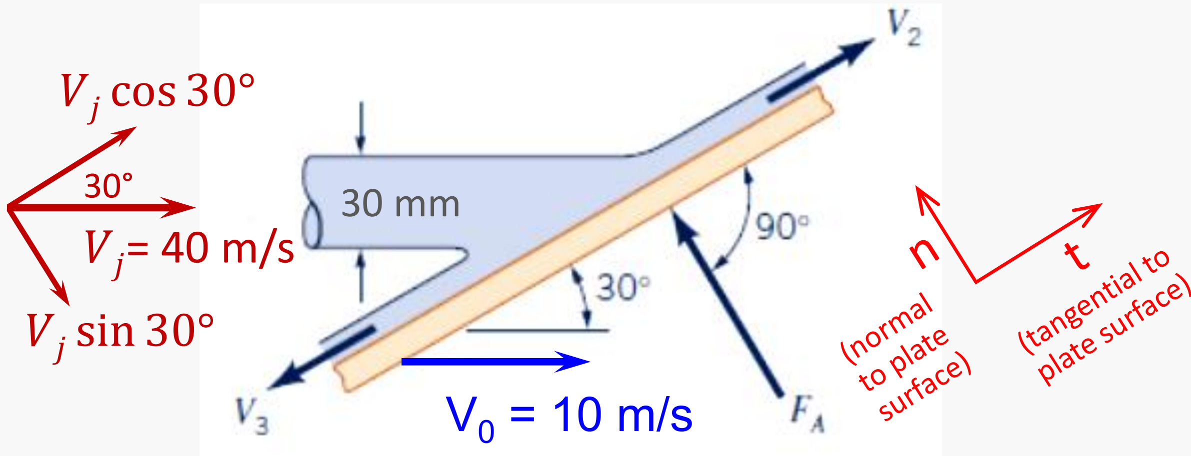
Ans. (b)

Problem # 12

(c) Control volume (CV) is moving to the right at velocity V_0 = 10 m/s

So, the velocity of water jet relative to the moving CV is

$V_r = V_1 - V_0$ = 40 m/s –10 m/s = 30 m/s



Steady flow momentum equation normal to the plate (n):

$$\sum F_n = (\dot{m}v)_{out} - (\dot{m}v)_{in}$$

$$\rightarrow F_A = 0 - \dot{m}_r(-v_r \sin 30°)$$

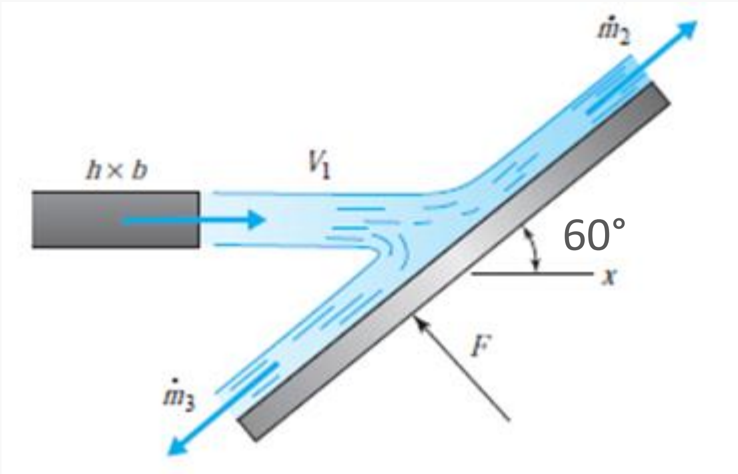
$$\rightarrow F_A = \left(\rho \frac{\pi}{4} D_j{}^2 v_r\right)(v_r \sin 30°)$$

F_A = 0.391 N   Ans. (c)

Problem # 13

Water flows from the rectangular jet as shown in Fig. Find the force *F* and the mass fluxes $\dot{m}_2$ and $\dot{m}_3$ if *b* = 20 mm, *h* = 40 cm, V_1 = 40 m/s.

If the plate moves to the left at 20 m/s, find the power required.



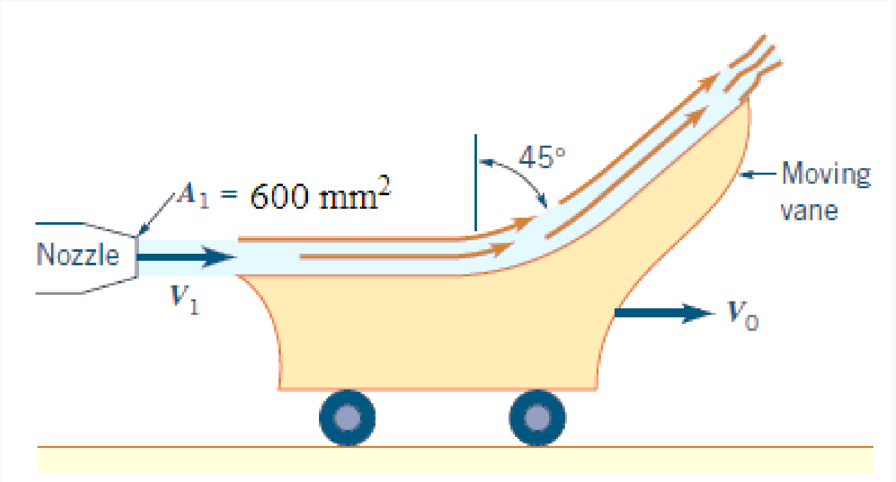
Ans.
F = 11.08 kN
$\dot{m}_2$ = 240 kg/s
$\dot{m}_3$ = 80 kg/s

$$F = \dot{m}_r (V_{1r})_n = 1000 \times 0.02 \times 0.4 \times (40+20)^2 \sin 60^\circ = 24\ 940\ \text{N}.$$

$$F_x = 24\ 940 \cos 30^\circ = 21\ 600\ \text{N}. \quad \therefore \dot{W} = 21\ 600 \times 20 = \underline{432\ 000\ \text{W}}.$$

Problem # 14

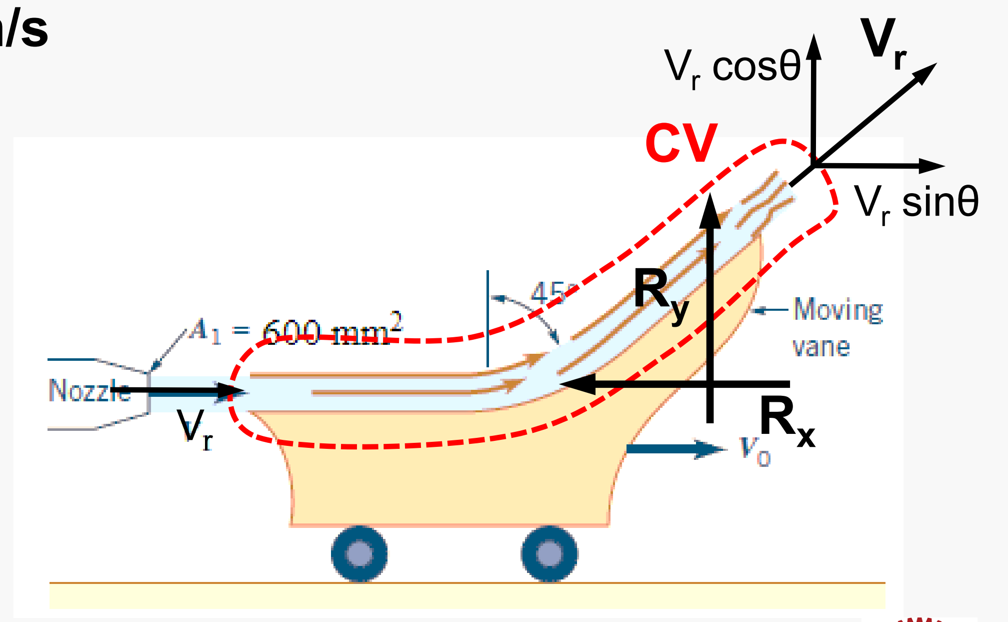
A vane on wheels moves with constant velocity V_0 when a stream of water having a nozzle exit velocity of V_1 is turned by the vane as indicated in figure. The speed of the water jet leaving the nozzle is 30 m/s, and the vane is moving to the right with a constant speed of 6 m/s.



Determine the magnitude and direction of the force exerted by the stream of water on the vane surface. *Consider ideal fluid flows and neglect the body force.*

Solution:

Control volume (CV) is moving to the right at velocity V_0 = 6 m/s



So, the velocity of water jet relative to the moving CV is

$$V_r = V_1 - V_0 = 30 \text{ m/s} - 6 \text{ m/s} = 24 \text{ m/s}$$

Problem 14

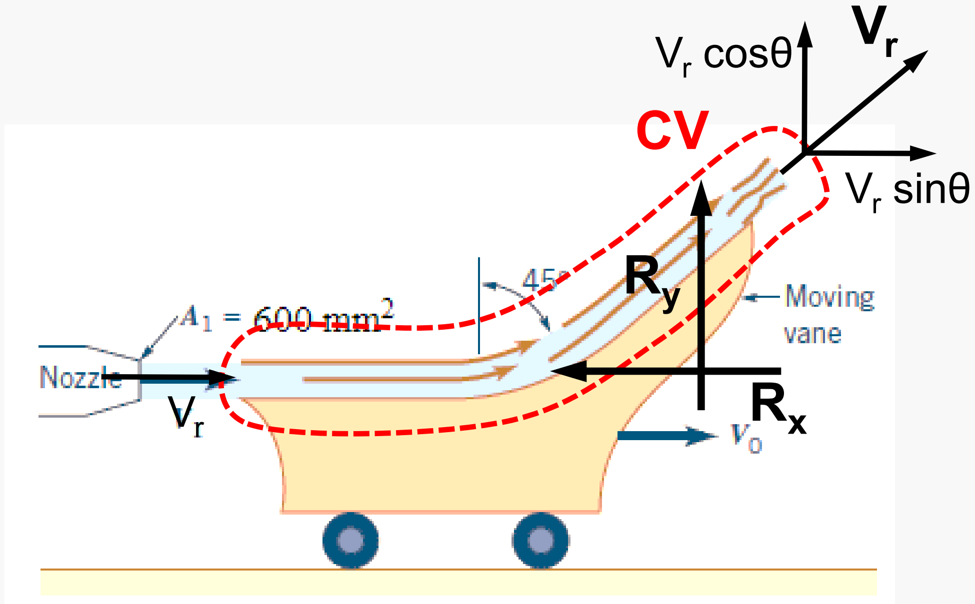
from steady flow momentum principle:

$$\boxed{\sum \vec{F}_{CV} = \int_{CS} \vec{\mathbf{V}}\rho\left(\vec{\mathbf{V}}\cdot\hat{\mathbf{n}}\right)dA}$$

$$\rightarrow x: \sum F_x = \left(\dot{m}V_x\right)_{out} - \left(\dot{m}V_x\right)_{in}$$

$$\Rightarrow -R_x = \dot{m}(V_{2x} - V_{1x})$$

$$\Rightarrow -R_x = \dot{m}(V_r \sin 45° - V_r)$$



$$\dot{m} = \rho_1 A_1 V_1 = \rho_1 A_1 V_r \qquad (\because \text{moving CV})$$

$$\Rightarrow \dot{m} = (1000)(600\times10^{-6})(24)$$

$$\therefore \dot{m} = 14.4 \text{ kg/s}$$

$$\therefore -R_x = (14.4)(24\sin 45° - 24)$$

$$\Rightarrow R_x = 101.2\,\text{N} \qquad \text{(to left)}$$

Problem 14

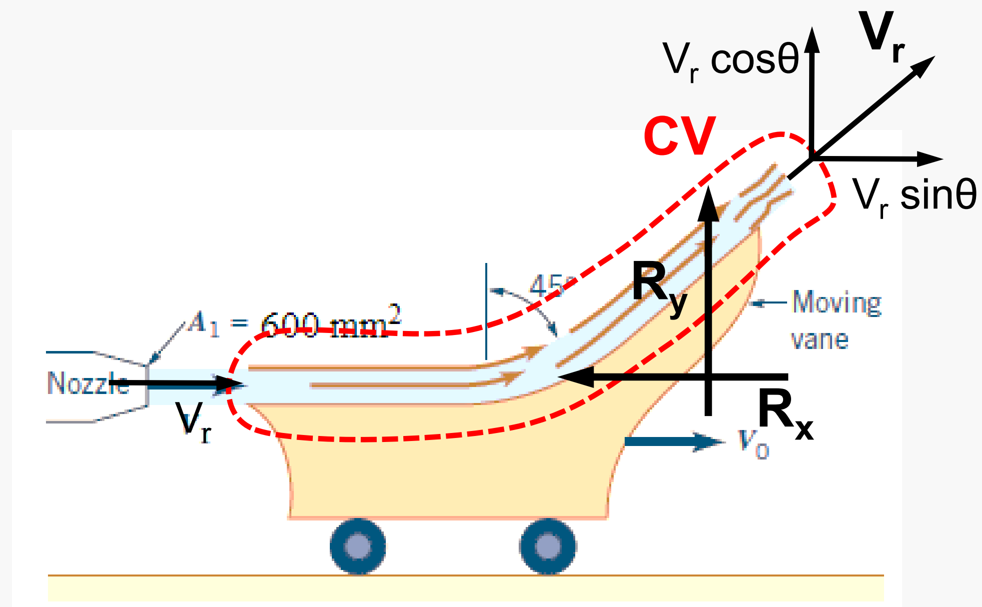
Now y-momentum equation:

$$\uparrow y: \ \sum R_y = \dot{m}\left(V_{2y} - V_{1y}\right)$$

$$\Rightarrow R_y = \dot{m}\left(V_r \cos 45° - 0\right)$$

$$\Rightarrow R_y = (14.4)(24 \cos 45°)$$

$$\therefore R_y = 244.3\,\text{N} \quad \text{(to upward)}$$



So, the magnitude of resultant force is

$$R = \sqrt{R_x^2 + R_y^2}$$

$$\Rightarrow R = \sqrt{101.2^2 + 244.4^2}$$

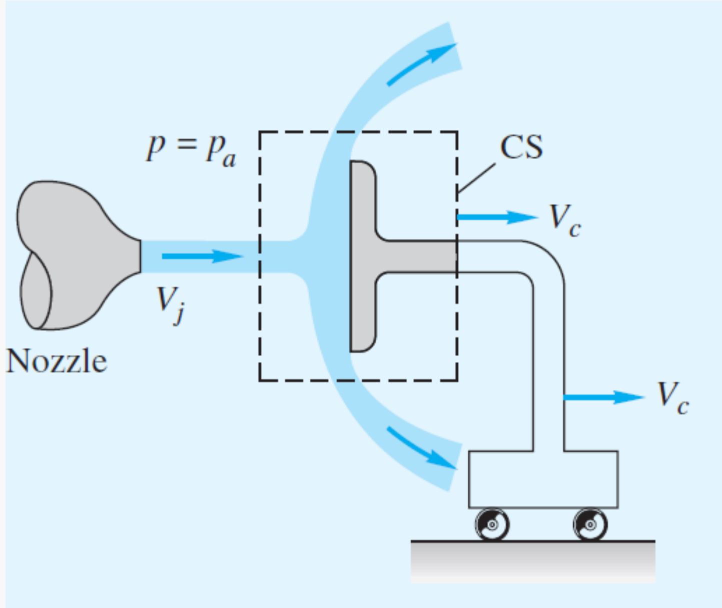
$$\Rightarrow R = 264.5 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \ 67.5° \ \text{ with} \left(-\text{ve}\right) x - \text{axis}$$

Problem # 15

EXAMPLE 3.9

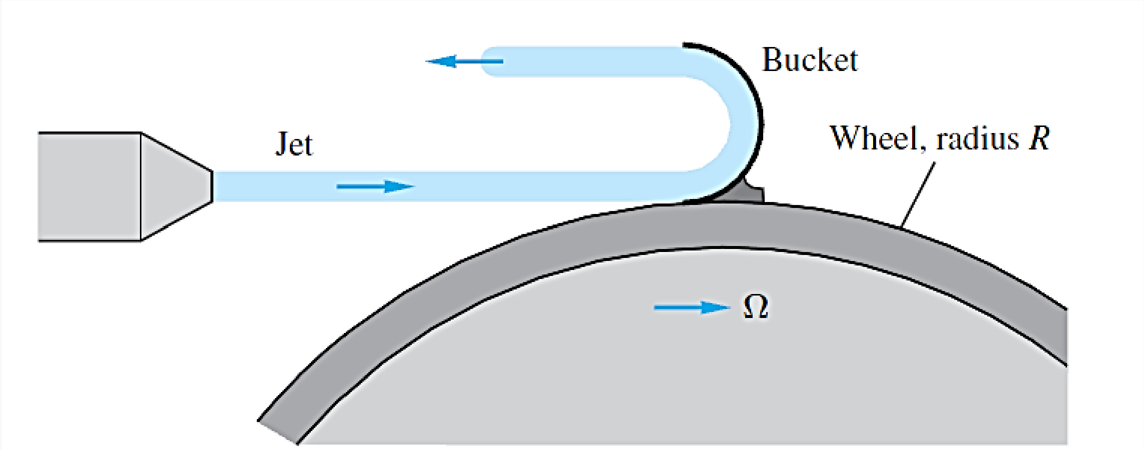
A water jet of velocity V_j impinges normal to a flat plate that moves to the right at velocity V_c, as shown in Fig. 3.9*a*. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m^3, the jet area is 3 cm^2, and V_j and V_c are 20 and 15 m/s, respectively. Neglect the weight of the jet and plate, and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.



Problem # 16

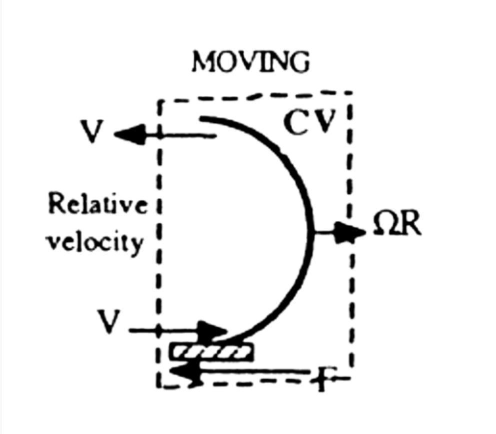
A liquid jet of velocity V_j and area A_j strikes a single 180° bucket on a turbine wheel rotating at angular velocity Ω , as in figure. Derive an expression for the power P delivered to this wheel at this instant as a function of the system parameters. At what angular velocity is the maximum power delivered?

How would your analysis differ if there were many, many buckets on the wheel, so that the jet was continually striking at least one bucket?



Solution:

Control volume (CV) is moving to the right at bucket velocity V = ΩR.



<u>First part:</u>

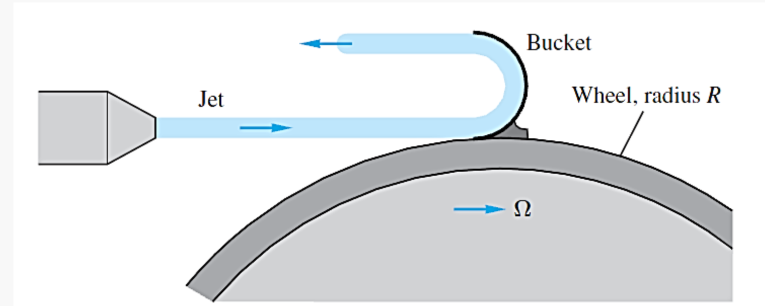
$$\rightarrow x: \sum F_x = \dot{m}\left(V_{x,out} - V_{x,in}\right)$$

$$\Rightarrow -F = \dot{m}\left[-\left(V_j - \Omega R\right) - \left(V_j - \Omega R\right)\right]$$

$$\Rightarrow -F = -2\dot{m}\left(V_j - \Omega R\right)$$

$$\therefore F = 2\dot{m}\left(V_j - \Omega R\right)$$

Problem # 16



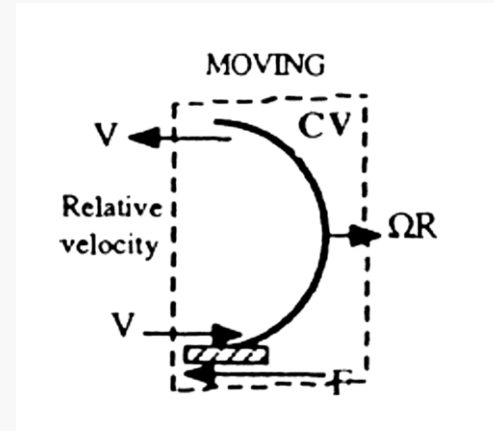
Now, mass flow rate is

$$\dot{m} = \rho A_j (V_j - \Omega R)$$

Then the force acting on the bucket becomes

$$F = 2\dot{m}\left(V_j - \Omega R\right) = 2\left[\rho A_j (V_j - \Omega R)\right]\left(V_j - \Omega R\right)$$

$$\Rightarrow F = 2\rho A_j \left(V_j - \Omega R\right)^2$$



Then the Power delivered to this wheel is

$$P = F \times \Omega R$$

$$\Rightarrow F = 2\rho A_j \Omega R \left(V_j - \Omega R\right)^2 \qquad \text{Ans.}$$

Maximum Power delivered can be obtained as:

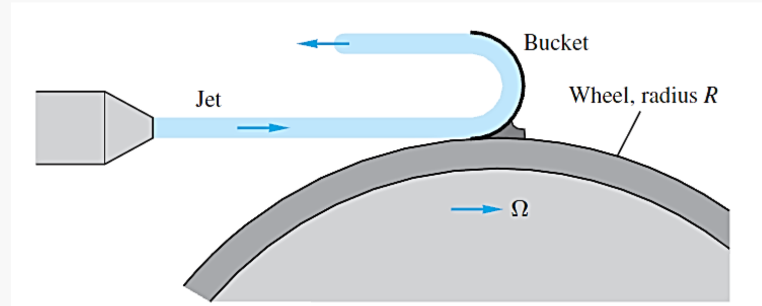
$$\frac{dP}{d\Omega} = 0$$

$$\Rightarrow \frac{d\left(2\rho A_j \Omega R \left(V_j - \Omega R\right)^2\right)}{d\Omega} = 0$$

$$\Rightarrow \Omega R = \frac{V_j}{3} \qquad \text{Ans.}$$

Problem # 16

Second part: Jet is continuously striking atleast one bucket



In this case, full jet mass will be available, and the flow rate comes as:

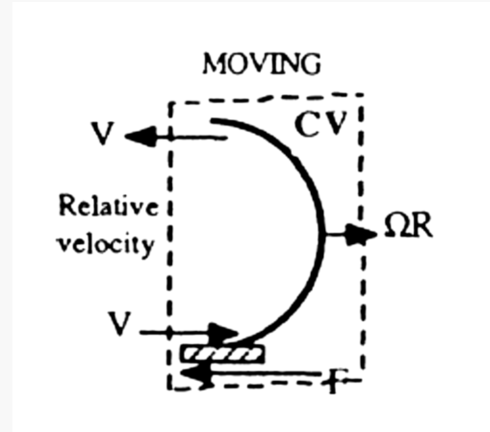
$$\dot{m} = \rho A_j V_j$$

Different from first part !!

Then the force acting on the bucket becomes

$$F = 2\dot{m}\left(V_j - \Omega R\right)$$

$$\Rightarrow F = 2\rho A_j V_j \left(V_j - \Omega R\right)$$



Then the Power delivered to this wheel is

$$P = F \times \Omega R$$

$$\Rightarrow F = 2\rho A_j V_j \Omega R\left(V_j - \Omega R\right) \qquad \text{Ans.}$$

Maximum Power delivered can be obtained as:

$$\frac{dP}{d\Omega} = 0$$

$$\Rightarrow \frac{d\left(2\rho A_j V_j \Omega R\left(V_j - \Omega R\right)\right)}{d\Omega} = 0$$

$$\Rightarrow \Omega R = \frac{V_j}{2} \qquad \text{Ans.}$$